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Viscous effects on transient long-wave propagation

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Using a perturbation approach and the Boussinesq approximation, we derive sets of depth-integrated continuity and momentum equations for transient long-wave propagation with viscous effects included. The fluid motion is assumed to be essentially irrotational, except in the bottom boundary layer. The resulting governing equations are differential–integral equations in terms of the depth-averaged horizontal velocity (or velocity evaluated at certain depth) and the free-surface displacement, in which the viscous terms are represented by convolution integrals. We show that the present theory recovers the well-known approximate damping rates for simple harmonic progressive waves and for a solitary wave. The relationship between the bottom stress and the depth-averaged velocity is discussed.

1. Introduction

In recent years, several phase-resolving and depth-integrated Boussinesq-type equations have been developed to simulate water-wave propagation from intermediate water depth to the surf zone (e.g. Liu 1994). These phase-resolving equations have been proposed as the 'wave driver' for calculating sediment-bedload transport fluxes and, in turn, morphological changes. Therefore, it is essential to have an accurate estimation of the bottom shear stress based on the calculated wave field above the bed. To include the viscous effects of the bottom boundary layer, traditionally a bottom-shear-stress term is added to the depth-integrated momentum equations and the bottom shear stress is then modelled as a function of the horizontal irrotational velocity on the bottom (also called the free-stream velocity herein) with an empirical coefficient. The simplest bottom-shear-stress model assumes that it is linearly proportional to the free-stream velocity. Thus, the shear stress is in phase with the free-stream velocity and has no memory of the history of the spatial variation of the free-stream velocity. It is well known that for a laminar boundary layer the phase difference between the bottom shear stress and the free-stream velocity is $\pi/4$ (see e.g. Mei 1983). This simple approach is inadequate for producing useful information on bottom shear stress under transient waves.

Byatt-Smith (1971) made an attempt to include the laminar viscous effects by obtaining the analytical solution for the rotational velocity in the linearized transient bottom boundary layer. Based on this analytical solution, Byatt-Smith obtained the bottom shear stress as a convolution integral in terms of the time derivative of the free-stream velocity, which was then included in the depth-averaged momentum equation. Byatt-Smith's formulation, which revealed the phase relation between the bottom shear stress and the free-stream velocity, is an improvement of the traditional approach mentioned above. However, the formulation is not rigorous enough in the sense that the order of accuracy of the governing equations used is not clear.

In this paper, we adopt and extend the perturbation approach outlined in Mei & Liu (1973) and Liu & Earickson (1983) to include the effects of a bottom boundary layer on the transient long-wave propagation. For simplicity, we have assumed that the viscosity is a constant. Therefore, the boundary layer is either laminar or turbulent with a constant eddy viscosity. It has been shown, in both theory and experiments, that the eddy viscosity is a constant in an oscillatory turbulent boundary layer when the bottom roughness is relatively large (van Doorn 1983; Nielsen 1992). Denoting ϵ as the parameter representing the magnitude of nonlinearity, μ^2 the frequency dispersion, and α the viscous effect, the Boussinesq approximation and the assumption that the viscous effects are slightly weaker, i.e. $O(\epsilon) \sim O(\mu^2)$ and $O(\alpha) \sim O(\epsilon^2) \sim O(\mu^4)$, are employed to derive a set of phase-resolving and depth-integrated continuity equation and a momentum equation. The effects of the bottom boundary layer are included through the vertical rotational velocity induced in the boundary layer. As explained in Mei & Liu (1973), the vertical velocity, working against the dynamic pressure, provides a mechanism for the transference of energy from the core region to the boundary layer. In the resulting governing equations, the viscous effects are represented by convolution integrals. Therefore, the viscous effects have a memory of the history of the spatial variation of the free-stream velocity. These equations are new and can be used to study various wave propagation problems in which the viscous effects might be significant. Finally, we present a formula for evaluating the leading-order bed stress in terms of the time history of the free-stream velocity.

As an illustration of the capability of the present formulation, we shall show that when the viscous effects are much weaker than the assumption proposed here, i.e. $O(\alpha) \ll O(\epsilon\mu^2)$, an additional perturbation expansion can be employed to find the evolution equation of the amplitude of the solitary wave due to viscous damping. The simplified analytical solution is the same as that found in Keulegan (1948) and Mei (1983). This implies that the new wave equations allow a more significant impact of the viscous boundary layer by including the higher-order terms.

2. Governing equations and boundary conditions

Consider a wave train with surface displacement $\zeta'(x', y', t')$ propagating in water of constant depth, h'. The wave train is characterized by a typical wave amplitude, a_0', a horizontal length scale, l_o', which is related to the magnitude of wavelength, and the time scale, $l_o'/\sqrt{gh'}$. The following dimensionless variables are introduced:

$$\left.\begin{aligned} &(x, y) = (x', y')/l_o', \quad z = z'/h', \quad t = \sqrt{gh'}t'/l_o', \\ &\zeta = \zeta'/a_0', \quad p = p'/\rho g a_0', \\ &(u, v) = (u', v')/\epsilon\sqrt{gh'}, \quad w = \mu w'/\epsilon\sqrt{gh'}, \end{aligned}\right\} \tag{2.1}$$

in which p' denotes the pressure, (u', v') the horizontal velocity components in the (x', y')-directions, w' the vertical velocity component in the z'-direction, ρ the fluid density, and g the gravitational acceleration. Two dimensionless parameters are introduced in (2.1):

$$\epsilon = a_0'/h', \quad \mu = h'/l_o'. \tag{2.2}$$

The dimensionless governing equations can be expressed as

$$\mu^2\nabla\cdot\boldsymbol{u} + \frac{\partial w}{\partial z} = 0, \tag{2.3}$$

$$\frac{\partial \boldsymbol{u}}{\partial t}+\epsilon \boldsymbol{u}\cdot\nabla\boldsymbol{u}+\frac{\epsilon}{\mu^2}w\frac{\partial \boldsymbol{u}}{\partial z}=-\nabla p+\alpha^2\left[\nabla^2\boldsymbol{u}+\frac{1}{\mu^2}\frac{\partial^2\boldsymbol{u}}{\partial z^2}\right], \tag{2.4}$$

$$\epsilon\frac{\partial w}{\partial t}+\epsilon^2\boldsymbol{u}\cdot\nabla w+\frac{\epsilon^2}{\mu^2}w\frac{\partial w}{\partial z}=-\epsilon\frac{\partial p}{\partial z}-1+\epsilon\alpha^2\left[\nabla^2 w+\frac{1}{\mu^2}\frac{\partial^2 w}{\partial z^2}\right], \tag{2.5}$$

in which

$$\alpha^2=\frac{\nu}{l_o'\sqrt{gh'}}, \tag{2.6}$$

with ν the constant eddy viscosity, which can be viewed as the inverse of Reynolds number. Note that in the following analysis the Boussinesq approximation, $O(\epsilon)\sim O(\mu^2)$, will be employed for weakly nonlinear and weakly dispersive waves. Also, we shall also require that $O(\alpha)\sim O(\epsilon^2)\sim O(\mu^4)$, which can be fulfilled in the following typical scenario. Consider a case where $O(\epsilon)\sim O(\mu^2)\sim 0.1$: if the water depth ($h'$) is 1 m and the eddy viscosity is $10^{-3}\,\mathrm{m^2\,s^{-1}}$ (which is about 1000 times the kinematic viscosity of water), the value of α is roughly 0.01, which is approximately the same as $O(\epsilon^2)$.

The dynamics of the flow problem can be presented in the following manner. The flow motions are essentially irrotational except in the boundary layers adjacent to the free surface, $z=\epsilon\zeta$, and the bottom, $z=-1$. It is well-known that in order to satisfy the no-slip boundary condition on the bottom, the leading order of magnitude of the horizontal rotational velocity component inside the bottom boundary layer is $O(1)$, while the horizontal rotational velocity component inside the free-surface boundary layer is weaker, $O(\alpha)$, since only the zero shear stress condition on the free surface is required (e.g. Mei & Liu 1973). From the continuity equation, a vertical velocity component of $O(\alpha)$ is generated inside the bottom boundary layer, which persists outside the boundary layer. Therefore, the irrotational flow in the core region must be corrected at order α.

Accordingly, we now introduce the following perturbation expansions:

$$\boldsymbol{u}=\nabla\Phi(\boldsymbol{x},z,t)+\boldsymbol{u}_0^r(\boldsymbol{x},z,t)+\alpha\boldsymbol{u}_1^r(\boldsymbol{x},z,t)+\cdots, \tag{2.7}$$

$$w=\frac{\partial\Phi}{\partial z}+\alpha\mu w_1^r+\cdots. \tag{2.8}$$

The velocity potential Φ has been introduced for the irrotational flow in the core region. Both the velocity potential and the rotational velocity components are further expanded in terms of ϵ and μ^2 in the following sections. The perturbation expansions are up to $O(\alpha)$.

2.1. *Bottom boundary layer analysis*

In this section, we shall focus on the flow inside the bottom boundary layer. Since the boundary layer thickness is of $O(\alpha)$, we introduce the stretched coordinate

$$\eta=\frac{z+1}{(\alpha/\mu)}. \tag{2.9}$$

The leading-order continuity equation for the rotational velocity in the bottom boundary layer becomes

$$\nabla\cdot\boldsymbol{u}_0^r+\frac{\partial w_1^r}{\partial\eta}=0. \tag{2.10}$$

The leading-order momentum equations can be expressed as

$$\frac{\partial \boldsymbol{u}_0^r}{\partial t} + \epsilon \left[\boldsymbol{u}_0^r \cdot \nabla \boldsymbol{u}_0^r + w_1^r \frac{\partial \boldsymbol{u}_0^r}{\partial \eta} \right] = \frac{\partial^2 \boldsymbol{u}_0^r}{\partial \eta^2}. \tag{2.11}$$

The dynamic pressure is an invariant across the boundary layer, i.e. $\partial p/\partial z = 0$, to the leading order.

The no-slip conditions on the bottom require that the rotational velocity satisfies the following boundary conditions:

$$\boldsymbol{u}_0^r = -\nabla \Phi, \quad \frac{\partial \Phi}{\partial z} = -\alpha \mu w_1^r, \quad \eta = 0. \tag{2.12}$$

At the outer edge of the boundary layer, $\eta \to \infty$, the horizontal rotational velocity components vanish,

$$\boldsymbol{u}_0^r, \quad w_1^r \to 0, \quad \eta \to \infty. \tag{2.13}$$

To find the leading-order solution for $\boldsymbol{u}_0^r$, subject to the boundary condition (2.12), the boundary layer equation (2.11) is first linearized:

$$\frac{\partial \boldsymbol{u}_0^r}{\partial t} = \frac{\partial^2 \boldsymbol{u}_0^r}{\partial \eta^2}. \tag{2.14}$$

The analytical solution for the two-point boundary-value problem is (Mei 1995)

$$\boldsymbol{u}_0^r(\boldsymbol{x}, \eta, t) = -\frac{\eta}{\sqrt{4\pi}} \int_0^t \frac{\nabla \Phi(\boldsymbol{x}, z=-1, T)}{\sqrt{(t-T)^3}} \mathrm{e}^{-\eta^2/4(t-T)} \, \mathrm{d}T. \tag{2.15}$$

From the continuity equation, (2.10), the vertical rotational velocity component can be obtained by integration,

$$w_1^r(\boldsymbol{x}, \eta, t) = -\int_\eta^\infty \mathrm{d}\eta \frac{\eta}{2\sqrt{\pi}} \int_0^t \frac{\nabla^2 \Phi(\boldsymbol{x}, z=-1, T)}{\sqrt{(t-T)^3}} \mathrm{e}^{-\eta^2/4(t-T)} \, \mathrm{d}T. \tag{2.16}$$

On the bottom, $\eta = 0$, the vertical rotational velocity takes the following form:

$$w_1^r = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\nabla^2 \Phi(\boldsymbol{x}, z=-1, T)}{\sqrt{(t-T)}} \, \mathrm{d}T. \tag{2.17}$$

The existence of this vertical boundary-layer velocity requires a correction to the irrotational core-region flow so that the no-flux condition at the bottom is satisfied up to $O(\alpha)$, i.e. (2.12). If the nonlinearity is considered in the boundary layer solutions, an $O(\epsilon)$ horizontal rotational velocity will be added to the boundary-layer solution, (2.15). Through the continuity equation this additional horizontal rotational velocity will generate $O(\alpha\mu\epsilon)$ vertical rotational velocity in the boundary layer, which in turn requires an additional correction to the irrotational core-region flow at $O(\alpha\mu\epsilon)$. However, since $O(\alpha\mu\epsilon) \sim O(\mu^7)$, this effect is not considered in the core-region flow analysis.

2.2. *Irrotational flows in the core region*

Now we shall turn our attention to the irrotational flows in the core region. In terms of the velocity potential as defined in (2.8), the continuity equation can be expressed as

$$\mu^2 \nabla^2 \Phi + \frac{\partial^2 \Phi}{\partial z^2} = 0, \quad -1 < z < \epsilon \zeta. \tag{2.18}$$

The dynamic and kinematic free-surface boundary conditions require

$$\mu^2\left(\frac{\partial \Phi}{\partial t}+\zeta\right)+\frac{1}{2}\epsilon\left[\mu^2|\nabla\Phi|^2+\left(\frac{\partial \Phi}{\partial z}\right)^2\right]=0, \quad z=\epsilon\zeta, \tag{2.19}$$

$$\mu^2\left[\frac{\partial \zeta}{\partial t}+\epsilon\nabla\Phi\cdot\nabla\zeta\right]=\frac{\partial \Phi}{\partial z}, \quad z=\epsilon\zeta, \tag{2.20}$$

in which the atmospheric pressure has been assumed to be a constant. The bottom boundary condition for the irrotational flow is the no-flux condition as given in (2.12). Substituting (2.17) into (2.12) yields, up to $O(\alpha)$,

$$\frac{\partial \Phi}{\partial z}=\frac{\alpha\mu}{\sqrt{\pi}}\int_0^t \frac{\nabla^2\Phi(\boldsymbol{x},z,T)}{\sqrt{(t-T)}}\,\mathrm{d}T, \quad z=-1. \tag{2.21}$$

Through the boundary-layer development the above bottom boundary condition provides a mechanism for including the viscous effect in the core-region flow. Because of the diffusion process in the boundary layer, the influence of viscosity is not instantaneous. In (2.21), the integrand is weighted by $(t-T)^{-1/2}$. Therefore, the effects of the boundary layer are cumulative, but are weighted in favour of the current time, $T=t$.

3. Boussinesq equations

In this section, we shall present the simplified governing equations for the irrotational flow by adopting the Boussinesq approximation, i.e. $O(\epsilon)\sim O(\mu^2)$. Moreover, we have further assumed that $O(\alpha)\sim O(\mu^4)$. The primary difference between the traditional Boussinesq equations and the present situation is that, in the traditional Boussinesq equations, the vertical velocity at the bottom is zero, while the present situation is not. On the other hand, in tsunami research, the bottom deformation due to an earthquake is often prescribed, and hence the vertical velocity is also given (e.g. Liu & Earickson 1983). Following Liu & Earickson's approach, we expand the potential function as a power series in the vertical coordinate,

$$\Phi(\boldsymbol{x},z,t)=\sum_{n=0}^{\infty}(z+1)^n\phi_n(\boldsymbol{x},t). \tag{3.1}$$

Substituting the expansion into the Laplace equation, (2.18), and the bottom boundary condition, (2.21), we obtain the following recursive relation:

$$\phi_{n+2}=\frac{-\mu^2\nabla^2\phi_n}{(n+1)(n+2)}, \quad n=0,1,2,\ldots, \tag{3.2}$$

with

$$\phi_1=\frac{\alpha\mu}{\sqrt{\pi}}\int_0^t \frac{\nabla^2\phi_0(\boldsymbol{x},T)}{\sqrt{(t-T)}}\,\mathrm{d}T. \tag{3.3}$$

In the traditional Boussinesq equations ϕ_1 vanishes, as do all the ϕ_n with odd n. Thus, using the recursive relation in the expansion, we obtain the potential function truncated up to $O(\mu^5)$:

$$\Phi=\phi_0+(z+1)\phi_1-\frac{\mu^2}{2}(z+1)^2\nabla^2\phi_0+\frac{\mu^4}{24}(z+1)^4\nabla^2\nabla^2\phi_0+O(\mu^6). \tag{3.4}$$

Defining the bottom horizontal velocity and the total depth as

$$\boldsymbol{u}_{\mathrm{b}} = \nabla\phi_0, \quad H = 1 + \epsilon\zeta, \tag{3.5}$$

the kinematic free-surface boundary condition, (2.20), becomes

$$\frac{1}{\epsilon}\frac{\partial H}{\partial t} + \nabla\cdot(H\boldsymbol{u}_{\mathrm{b}}) - \frac{\mu^2}{6}\nabla^2(\nabla\cdot\boldsymbol{u}_{\mathrm{b}}) - \frac{1}{\mu^2}\phi_1 = O(\mu^4), \tag{3.6}$$

where ϕ_1 can be modified, from (3.3), to

$$\phi_1 = \frac{\alpha\mu}{\sqrt{\pi}}\int_0^t \frac{\nabla\cdot\boldsymbol{u}_{\mathrm{b}}}{\sqrt{(t-T)}}\,\mathrm{d}T. \tag{3.7}$$

We reiterate that the Boussinesq assumption, i.e. $O(\mu^2)\sim O(\epsilon)$, and the assumption that $O(\alpha)\sim O(\epsilon^2)$ have been employed.

Similarly, the dynamic free-surface boundary condition, (2.19), can be expressed in terms of H and $\boldsymbol{u}_{\mathrm{b}}$ as

$$\frac{\partial\boldsymbol{u}_{\mathrm{b}}}{\partial t} + \epsilon\boldsymbol{u}_{\mathrm{b}}\cdot\nabla\boldsymbol{u}_{\mathrm{b}} + \frac{1}{\epsilon}\nabla H - \frac{\mu^2}{2}\nabla\left[\nabla\cdot\frac{\partial\boldsymbol{u}_{\mathrm{b}}}{\partial t}\right] = O(\mu^4). \tag{3.8}$$

Equations (3.6) and (3.8) constitute the Boussinesq-type equations in terms of the bottom velocity, $\boldsymbol{u}_{\mathrm{b}}$, and the total depth, H, with the effects of a viscous bottom boundary layer included.

Traditionally, Boussinesq equations are expressed in terms of the depth-averaged horizontal velocity. By definition, the depth-averaged velocity is given as

$$\overline{\boldsymbol{u}} = \frac{1}{H}\int_{-1}^{\epsilon\zeta}\nabla\Phi\,\mathrm{d}z = \boldsymbol{u}_{\mathrm{b}} - \frac{\mu^2}{6}H^2\nabla^2\boldsymbol{u}_{\mathrm{b}} + O(\mu^4). \tag{3.9}$$

Substituting the above equation into to (3.6) and (3.8), we obtain

$$\frac{1}{\epsilon}\frac{\partial H}{\partial t} + \nabla\cdot(H\overline{\boldsymbol{u}}) - \frac{\alpha}{\mu\sqrt{\pi}}\int_0^t\frac{\nabla\cdot\overline{\boldsymbol{u}}}{\sqrt{t-T}}\,\mathrm{d}T = O(\mu^4), \tag{3.10}$$

$$\frac{\partial\overline{\boldsymbol{u}}}{\partial t} + \epsilon\overline{\boldsymbol{u}}\cdot\nabla\overline{\boldsymbol{u}} + \frac{1}{\epsilon}\nabla H - \frac{\mu^2}{3}\nabla\nabla\cdot\left(\frac{\partial\overline{\boldsymbol{u}}}{\partial t}\right) = O(\mu^4). \tag{3.11}$$

If the viscous effects are ignored, i.e. $\alpha\to 0$, (3.10) and (3.11) reduce to the conventional Boussinesq equations. It is clear that the leading-order viscous effects are due to the boundary layer displacement in the mass balance.

It is well known that the frequency–dispersion characteristics of the Boussinesq equations, expressed in terms of the horizontal velocity evaluated at a certain elevation, $z = z_\alpha = -0.521$, are better than those in terms of the bottom velocity and the depth-averaged velocity (Nwogu 1993). (In other words, these Boussinesq equations can be applied to deeper water or shorter wavelength.) Therefore, in the Appendix we also present the Boussinesq equations in terms of the velocity evaluated at z_α with consideration of bottom boundary layer effects.

3.1. *Bed shear stress*

By definition, the shear stress in the boundary layer in dimensionless form is

$$\boldsymbol{\tau} = \frac{\partial\boldsymbol{u}_0^r}{\partial\eta}. \tag{3.12}$$

Therefore, from (2.15), the leading-order bed shear stress, at $\eta = 0$, can be expressed as

$$\boldsymbol{\tau}_{\mathrm{b}} = -\frac{1}{2\sqrt{\pi}} \int_0^t \frac{\overline{\boldsymbol{u}}}{\sqrt{(t-T)^3}}\, \mathrm{d}T. \tag{3.13}$$

It is clear that the bottom shear stress depends not only on the depth-averaged horizontal velocity at the current time, but also on the horizontal velocity in the past. The relative importance is weighted by the function $(t-T)^{-3/2}$ for $0 < T < t$.

3.2. *One-dimensional cases*

To illustrate that the present formulation recovers the existing theories on the viscous damping of simple harmonic waves and solitary waves, we shall consider one-dimensional problems in this section. Thus, the continuity and momentum equations become

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}[(1+\epsilon\zeta)\overline{u}] - \frac{\alpha}{\mu}\frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial \overline{u}}{\partial x}\frac{1}{\sqrt{t-T}}\, \mathrm{d}T = 0, \tag{3.14}$$

$$\frac{\partial \overline{u}}{\partial t} + \epsilon\overline{u}\frac{\partial \overline{u}}{\partial x} + \frac{\partial \zeta}{\partial x} - \frac{\mu^2}{3}\frac{\partial^3 \overline{u}}{\partial x^2 \partial t} = 0. \tag{3.15}$$

3.2.1. *Viscous damping of linear progressive waves*

For linear progressive waves, (3.14) and (3.15) can be further simplified to

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \overline{u}}{\partial x} - \frac{\alpha}{\mu}\frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial \overline{u}}{\partial x}\frac{1}{\sqrt{t-T}}\, \mathrm{d}T = 0, \tag{3.16}$$

$$\frac{\partial \overline{u}}{\partial t} + \frac{\partial \zeta}{\partial x} = 0. \tag{3.17}$$

Introducing the moving coordinate

$$\sigma = x - t, \quad \xi = \left(\frac{\alpha}{\mu}\right) t, \tag{3.18}$$

into (3.16) and (3.17) and summing the resulting equations, we obtain

$$\frac{\partial \zeta}{\partial \xi} = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{\partial \zeta}{\partial x}\frac{1}{\sqrt{t-T}}\, \mathrm{d}T. \tag{3.19}$$

Note that $\zeta = \overline{u}$ has been used as a leading-order approximation. Because of the viscous damping, the free-surface displacement can be represented as:

$$\zeta = a(\xi)\mathrm{e}^{\mathrm{i}\sigma}. \tag{3.20}$$

Substituting the solution form, (3.20), into (3.16), we obtain

$$\frac{\partial a}{\partial \xi}\mathrm{e}^{\mathrm{i}\sigma} = \left[\frac{\mathrm{i}}{2\sqrt{\pi}} \int_0^{\xi/(\alpha/\mu)} \frac{\mathrm{e}^{\mathrm{i}(x-T)}}{\sqrt{t-T}}\, \mathrm{d}T\right] a(\xi). \tag{3.21}$$

Since the parameter α/μ is small, the upper limit of the above integral is essentially infinity. Thus, with a simple substitution of $\psi = t - T$, the right-hand side of (3.21) can be integrated analytically in terms of the Fresnel integrals as follows:

$$\text{R.H.S. of (3.21)} = \left[\frac{\mathrm{i}}{2\sqrt{\pi}}\mathrm{e}^{\mathrm{i}\sigma} \int_0^\infty \frac{\mathrm{e}^{\mathrm{i}\psi}}{\sqrt{\psi}}\, \mathrm{d}\psi\right] a(\xi) = a(\xi)\tfrac{1}{2}\mathrm{e}^{\mathrm{i}\sigma}\mathrm{e}^{-\mathrm{i}\pi/4}.$$

Substituting the above equation back into (3.21), and introducing

$$a = a_0 e^{i\beta\xi}, \quad \beta = \beta_r + i\beta_i, \tag{3.22}$$

where β_i denotes the damping rate, we find

$$\beta_r = \beta_i = \frac{1}{2\sqrt{2}} \tag{3.23}$$

which is exactly the same as the result obtained by Mei & Liu (1973) (see also Mei 1983).

3.2.2. *Viscous damping of solitary waves*

A similar analysis can be carried out for calculating the viscous damping of solitary waves. Following the approach outlined in the previous section, we can combine (3.14) and (3.15) in the moving frame, $\sigma = x - t$, as

$$\frac{\partial \zeta}{\partial \overline{\xi}} + \frac{3}{2}\zeta\frac{\partial \zeta}{\partial \sigma} + \frac{1}{6}\frac{\mu^2}{\epsilon}\frac{\partial^3 \zeta}{\partial \sigma^3} = \frac{\alpha}{\mu\epsilon}\frac{1}{2\sqrt{\pi}}\int_0^t \frac{\partial \zeta}{\partial x}\frac{1}{\sqrt{t-T}}\,\mathrm{d}T, \tag{3.24}$$

in which the slow time variable is defined as $\overline{\xi} = \epsilon t$. Without the damping effect, i.e. $\alpha \to 0$, the solitary wave solution can be written as

$$\zeta = a(\overline{\xi})\mathrm{sech}^2\left[\frac{\sqrt{3a}}{2}\left(\sigma - \frac{a}{2}\overline{\xi}\right)\right]. \tag{3.25}$$

Thus, with the viscous damping, we introduce the perturbation solution as follows:

$$\zeta = \zeta_0(\rho, \xi) + \delta\zeta_1(\rho, \xi) + \cdots, \tag{3.26}$$

where

$$\rho = \sigma - \frac{1}{2\delta}\int^{\xi} a(\xi')\,\mathrm{d}\xi', \quad \xi = \delta\overline{\xi}, \quad \delta = \frac{\alpha}{\mu\epsilon}. \tag{3.27}$$

Substituting of (3.26) into (3.24), and collecting terms at the same order, we obtain the following equations for the first two orders in δ:

$$\mathrm{L}_0\zeta_0 = \frac{\partial}{\partial \rho}\left[-\frac{a}{2} + \frac{3}{4}\zeta_0 + \frac{1}{6}\frac{\mu^2}{\epsilon}\frac{\partial^2}{\partial \rho^2}\right]\zeta_0 = 0, \tag{3.28}$$

$$\mathrm{L}_1\zeta_1 = \frac{\partial}{\partial \rho}\left[-\frac{a}{2} + \frac{3}{2}\zeta_0 + \frac{1}{6}\frac{\mu^2}{\epsilon}\frac{\partial^2}{\partial \rho^2}\right]\zeta_1 = \frac{1}{2\sqrt{\pi}}\int_0^t \frac{\partial \zeta_0}{\partial x}\frac{1}{\sqrt{t-T}}\,\mathrm{d}T - \frac{\partial \zeta_0}{\partial \xi}, \tag{3.29}$$

where L_0 and L_1 are adjoint operators of each other (Ott & Sudan 1970), i.e.

$$\int_{-\infty}^{\infty}(\zeta_0\mathrm{L}_1\zeta_1 - \zeta_1\mathrm{L}_0\zeta_0)\,\mathrm{d}\rho = 0. \tag{3.30}$$

Clearly the solution for the leading-order equation is just the solitary wave solution,

$$\zeta_0 = a(\xi)\mathrm{sech}^2\left[\frac{\sqrt{3a}}{2}\rho\right]. \tag{3.31}$$

Equation (3.30) provides a solvability condition for ζ_1:

$$\int_{-\infty}^{\infty}\zeta_0\left(\frac{1}{2\sqrt{\pi}}\int_0^t \frac{\partial \zeta_0}{\partial x}\frac{1}{\sqrt{t-T}}\,\mathrm{d}T - \frac{\partial \zeta_0}{\partial \xi}\right)\mathrm{d}\rho = 0. \tag{3.32}$$

The items in the integrand of the above integral can be expressed explicitly as

$$-\frac{\partial \zeta_0}{\partial \xi} = -\frac{\mathrm{d}a}{\mathrm{d}\xi}\mathrm{sech}^2\left(\frac{\sqrt{3a}}{2}\rho\right)\left[1 - \frac{\sqrt{3a}}{2}\rho\tanh\left(\frac{\sqrt{3a}}{2}\rho\right)\right],$$

$$\frac{1}{2\sqrt{\pi}}\int_0^t \frac{\partial \zeta_0}{\partial x}\frac{1}{\sqrt{t-T}}\,\mathrm{d}T = -\sqrt{\frac{3}{\pi}}a^{3/2}\int_0^\infty \frac{1}{\sqrt{S}}\mathrm{sech}^2\left[\frac{\sqrt{3a}}{2}(\rho+S)\right]\tanh\left[\frac{\sqrt{3a}}{2}(\rho+S)\right]\mathrm{d}S.$$

Thus, substituting the above equations and (3.31) into the solvability condition, (3.32), we find

$$\frac{\mathrm{d}a}{\mathrm{d}\xi} = -a^{5/4}\sqrt{\frac{2\sqrt{3}}{\pi}}\int_{-\infty}^{\infty}\mathrm{d}r\,\mathrm{sech}^2(r)\int_0^\infty \mathrm{sech}^2(r+S^2)\tanh(r+S^2)\,\mathrm{d}S, \tag{3.33}$$

which is exactly the same as the equation derived in Mei (1983). As mentioned in Mei, the double integral on the right-hand side of (3.33) has been evaluated by Keulegan (1948) and is approximately π^{-1}. Hence the viscous damping rate for solitary wave can be expressed as

$$1 - a^{-1/4} = -0.0836\xi. \tag{3.34}$$

4. Concluding remarks

A set of two-dimensional depth-averaged continuity and momentum equations including consideration of bottom boundary layer effects has been derived, (3.10) and (3.11). The viscous effects appear in both continuity and momentum equations in the form of convolution integrals. These equations can also be expressed in terms of the horizontal velocity at any elevation (see the Appendix). Several restrictions are imposed in the derivation. Some of them, such as the constant depth assumption, can be removed relatively easily.

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Appendix

In this Appendix, we present the Boussinesq type of depth-integrated continuity and momentum equations, written in terms of the velocity evaluated at elevation $z = z_\alpha$ and the free-surface displacement. Defining

$$\Phi_\alpha = \Phi(\boldsymbol{x}, z = z_\alpha, t) \tag{A 1}$$

as the velocity potential evaluated at the elevation $z = z_\alpha$, the velocity potential can be expressed in terms of Φ_α as

$$\Phi = \Phi_\alpha + (z - z_\alpha)\phi_1 - \frac{\mu^2}{2}\left[\left(z^2 - z_\alpha^2\right) + 2(z - z_\alpha)\right]\nabla^2\Phi_\alpha + \frac{\mu^4}{24}\left[\left(z^4 - z_\alpha^4\right) + 4\left(z^3 - z_\alpha^3\right) + 6\left(z^2 - z_\alpha^2\right) + 4(z - z_\alpha)\right]\nabla^2\nabla^2\Phi_\alpha + O(\mu^6). \tag{A 2}$$

Following the procedures presented above, we find the following governing equations:

$$\frac{1}{\epsilon}\frac{\partial H}{\partial t} + \nabla\cdot(H\boldsymbol{u}_\alpha) + \frac{\mu^2}{2}\nabla\cdot\left[\left(z_\alpha^2 + 2z_\alpha + \tfrac{2}{3}\right)\nabla(\nabla\cdot\boldsymbol{u}_\alpha)\right] - \frac{1}{\mu^2}\phi_1 = O(\mu^4), \tag{A 3}$$

$$\frac{\partial \boldsymbol{u}_\alpha}{\partial t} + \epsilon\boldsymbol{u}_\alpha\cdot\nabla\boldsymbol{u}_\alpha + \frac{1}{\epsilon}\nabla H + \mu^2\left(z_\alpha + \frac{z_\alpha^2}{2}\right)\nabla\left(\nabla\cdot\frac{\partial \boldsymbol{u}_\alpha}{\partial t}\right) = O(\mu^4), \tag{A 4}$$

in which

$$\phi_1 = \frac{\alpha\mu}{\sqrt{\pi}} \int_0^t \frac{\nabla \cdot \boldsymbol{u}_\alpha}{\sqrt{(t-T)}}\, \mathrm{d}T. \tag{A 5}$$

If the boundary-layer effects are ignored, (A 3)–(A 5) reduce to those originally derived by Nwogu (1993). Moreover, the Boussinesq equation written in terms of the bottom velocity, $\boldsymbol{u}_b$, (3.6) and (3.8), can be recovered from by applying $z_\alpha = -1$.

REFERENCES

Byatt-Smith, J. G. B. 1971 The effect of laminar viscosity on the solution of the undular bore. *J. Fluid Mech.* **48**, 33–40.

van Doorn, T. 1983 Computations and comparisons with experiments of the bottom boundary layer in an oscillatory flow. *TOW-Rep.* M 1562-2. Delft Hydraulics Laboratory.

Keulegan, G. H. 1948 Gradual damping of solitary wave. *J. Res. Natl Bur. Stand.* **40**, 607–614.

Liu, P. L.-F. 1994 Model equations for wave propagation from deep to shallow water. In *Advances in Coastal Engineering* (ed. P. L.-F. Liu), vol. 1, pp. 125–157. World Scientific.

Liu, P. L.-F. & Earickson, J. 1983 A numerical model for tsunami generation and propagation. In *Tsunamis: Their Science and Engineering* (ed. J. Iida & T. Iwasaki), pp. 227–240. Harpenden: Terra Science.

Mei, C. C. 1983 *The Applied Dynamics of Ocean Surface Waves*. John Wiley & Sons.

Mei, C. C. 1995 *Mathematical Analysis in Engineering*. Cambridge University Press.

Mei, C. C. & Liu, P. L.-F. 1973 The damping of surface gravity waves in a bounded liquid. *J. Fluid Mech.* **59**, 239–256.

Nielsen, P. 1992 *Coastal Boundary Layers and Sediment Transport*. World Scientific.

Nwogu, O. 1993 Alternative form of Boussinesq equations for nearshore wave propagation. *J. Waterway, Port, Coast. Ocean Engng* **119**, 618–638.

Ott, E. & Sudan, R. N. 1970 Damping of solitary waves. *Phys. Fluids* **13**, 1432.